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Self-adjoint extensions and SUSY breaking in supersymmetric quantum mechanics

H Falomir and P A G Pisani

IFLP (CONICET)—Departamento de Física, Fac. de Ciencias Exactas, UNLP, CC 67,
(1900) La Plata, Argentina

E-mail: falomir@fisica.unlp.edu.ar and pisani@fisica.unlp.edu.ar

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Abstract

We consider the self-adjoint extensions (SAE) of the symmetric supercharges and Hamiltonian for a model of SUSY quantum mechanics in \mathbb{R}^+ with a singular superpotential. We show that only for two particular SAE, whose domains are scale invariant, the algebra of $N = 2$ SUSY is realized, one with manifest SUSY and the other with spontaneously broken SUSY. Otherwise, only the $N = 1$ SUSY algebra is obtained, with spontaneously broken SUSY and non-degenerate energy spectrum.

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1. Introduction

Supersymmetry (SUSY) [1–6, 8–11] gives desirable features to quantum field theories, such as an improved ultraviolet behaviour, but also predicts superpartner states with degenerate mass which are not observed experimentally. Therefore, this symmetry is expected to be spontaneously (dynamically) broken.

Several schemes have been developed to try to solve the SUSY breaking problem, including the idea of non-perturbative breaking by instantons. In this context, the simplest case of SUSY quantum mechanics (SUSYQM) was introduced by Witten [8] and Cooper and Freedman [10].

When considering these models, several authors have suggested that singular potentials could break SUSY through nonstandard mechanisms, being responsible for non-degeneracy of energy levels and negative energy eigenstates [12–17].

In particular, Jevicki and Rodrigues [12] have considered the singular superpotential $W(x) = g/x - x$. Based on the square integrable solutions of a differential operator related to

the Hamiltonian of this system [18] they concluded that, for a certain range of the parameter g , SUSY is broken with a negative energy ground state.

However, they have not considered if all these functions correspond to eigenvectors of a unique self-adjoint Hamiltonian. As is well known, the quantum dynamics is given by a unitary group, and it follows from Stone's theorem [19] that the Hamiltonian, which is the infinitesimal generator of this group, must be self-adjoint.

Later, Das and Pernice [20] have reconsidered this problem in the framework of a SUSY preserving regularization of the singular superpotential, finding that SUSY is recovered manifestly at the end, when the regularization is removed. They conclude that SUSY is robust at short distances (high energies), and the singularities that occur in quantum-mechanical models are unlikely to break SUSY.

In the present paper, we would like to address this controversial subject by studying the self-adjoint extensions of the Hamiltonian defined by the singular superpotential $W(x) = g/x - x$ with $x \in \mathbb{R}^+$. This will be done by studying the self-adjoint extensions of the symmetric supercharges, and by considering the possibility of realizing the algebra of SUSY in a dense subspace of the Hilbert space.

We will show that there is a range of values of g for which the supercharges admit a one-parameter family of self-adjoint extensions, corresponding to a one-parameter family of self-adjoint extensions of the Hamiltonian. We will show that only for two particular self-adjoint extensions, whose domains are scale invariant, the algebra of $N = 2$ SUSY can be realized, one with manifest SUSY and the other with spontaneously broken SUSY. For other values of this continuous parameter, only the $N = 1$ SUSY algebra is obtained, with spontaneously broken SUSY and non-degenerate energy spectrum.

We should mention that self-adjoint extensions of supercharges and Hamiltonian for the SUSYQM of the free particle with a point singularity in the line and the circle have been considered in [21–24], where $N = 1, 2$ realizations of SUSY are described. They have also been considered in the framework of the Landau Hamiltonian for two-dimensional particles in nontrivial topologies in [25] (see also [26]).

Let us remark that, given a superpotential $W(x)$, one gets a *formal* expression for the Hamiltonian (and also for the supercharges) as a symmetric differential operator H defined on a subspace of sufficiently smooth square-integrable functions. The theory of *deficiency indices* of von Neumann [19] gives the basic criterion for the existence of *self-adjoint extensions* of this operator. In the case where there is only one self-adjoint extension, H is *essentially self-adjoint* and its *closure* [19] represents the true Hamiltonian of the system. But if there are several self-adjoint extensions of H , they usually differ by the *physics* they describe. In this case, the selection of a Hamiltonian among the self-adjoint extensions of H is not just a mathematical technicality. Rather, additional physical information is required to select the correct one, which describes the true properties of the system.

The structure of the paper is as follows: in the next section we present the problem to solve. In section 3 we study the adjoint operator of the supercharge, whose properties are needed to determine the supercharge self-adjoint extensions. This is done in section 4, where the self-adjoint extensions of the Hamiltonian are also determined. In section 5 we consider the possibility of realizing the algebra of the supersymmetry on the Hamiltonian domain of definition, and state our conclusions. In appendix A we treat some technicalities related to the closure of the symmetric supercharge and in appendix B we consider the graded partition function and the Witten index of the Hamiltonian, and the spectral asymmetry of the supercharge.

2. Setting of the problem

The Hamiltonian of a supersymmetric one-dimensional system can be written as

$$H = \{Q, \tilde{Q}\}_+, \quad (2.1)$$

where the supercharges

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} 0 & \tilde{A} \\ 0 & 0 \end{pmatrix} \quad (2.2)$$

are nilpotent operators,

$$Q^2 = \tilde{Q}^2 = 0, \quad (2.3)$$

which commute with the Hamiltonian.

In equation (2.2),

$$A = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + W(x) \right) \quad \text{and} \quad \tilde{A} = \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + W(x) \right) \quad (2.4)$$

are differential operators defined on a suitable dense subspace of functions where the necessary compositions of operators in equations (2.1) and (2.3) are well defined, and $W(x)$ is the superpotential.

In this section we will consider a quantum-mechanical system living in the half-line \mathbf{R}^+ , subject to a superpotential given by

$$W(x) = \frac{g}{x} - x \quad (2.5)$$

for $x > 0$ and g real. The two differential operators defined in (2.4) take the form

$$A = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + \frac{g}{x} - x \right), \quad (2.6)$$

$$\tilde{A} = \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + \frac{g}{x} - x \right). \quad (2.7)$$

Let us now introduce an operator Q_+ , defined on the dense subspace of (two component) functions with continuous derivatives of all order and compact support not containing the origin, $\mathcal{D}(Q_+) = C_0^\infty(\mathbf{R}^+ \setminus \{0\})$, over which its action is given by

$$Q_+ \Psi = \begin{pmatrix} 0 & \tilde{A} \\ A & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \quad (2.8)$$

Note that, within this domain, Q_+ can be identified with

$$Q_+ = \tilde{Q} + Q, \quad (2.9)$$

while its square (which is well defined) satisfies

$$Q_+^2 = \{Q, \tilde{Q}\}_+ = H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \quad (2.10)$$

where H is the Hamiltonian of the system, with $H_+ = \tilde{A}A$ and $H_- = A\tilde{A}$.

It can easily be verified that Q_+ so defined is a symmetric operator, but it is neither self-adjoint nor even closed. Consequently, we must look for the self-adjoint extensions of Q_+ .

Within the same domain, a linearly independent combination of supercharges leads to the operator

$$Q_- = i(\tilde{Q} - Q), \quad (2.11)$$

which is also symmetric and satisfies that $Q_-^2 = H$, and $\{Q_+, Q_-\}_+ = 0$. Since it can be obtained from Q_+ through a unitary transformation given by

$$Q_- = e^{i\sigma_3\pi/4} Q_+ e^{-i\sigma_3\pi/4}, \quad \text{with } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.12)$$

the following analysis will be carried out only for Q_+ , and it will extend immediately to Q_- .

Note that, given a self-adjoint extension of Q_+ (which, in particular, is a closed and densely defined operator [19]), its *square* gives the corresponding self-adjoint extension of the Hamiltonian H in equation (2.10), by virtue of a theorem due to von Neumann¹.

The first step in the construction of the self-adjoint extensions of Q_+ consists in the determination of its adjoint, Q_+^\dagger , which will be done in the next section.

3. The adjoint operator Q_+^\dagger

In this section we will determine the domain of definition of Q_+^\dagger , and its spectrum. In particular, we are interested in the *deficiency subspaces* [19] of Q_+ (the null subspaces of $(Q_+^\dagger \mp i)$),

$$\mathcal{K}_\pm := \text{Ker}(Q_+^\dagger \mp i), \quad (3.1)$$

which determine the self-adjoint extensions of Q_+ .

3.1. Domain of Q_+^\dagger

A (two-component) function Φ belongs to the domain of Q_+^\dagger ,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathcal{D}(Q_+^\dagger) \subset \mathbf{L}_2(\mathbb{R}^+), \quad (3.2)$$

if $(\Phi, Q_+\Psi)$ is a linear continuous functional of Ψ , for $\Psi \in \mathcal{D}(Q_+)$. This requires the existence of a function

$$\Omega = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \in \mathbf{L}_2(\mathbb{R}^+) \quad (3.3)$$

such that

$$(\Phi, Q_+\Psi) = (\Omega, \Psi), \quad \forall \Psi \in \mathcal{D}(Q_+). \quad (3.4)$$

Such Ω is uniquely determined, since $\mathcal{D}(Q_+)$ is a dense subspace. Then, for each $\Phi \in \mathcal{D}(Q_+^\dagger)$, the action of Q_+^\dagger is defined by $Q_+^\dagger\Phi := \Omega$. Note that $\mathcal{D}(Q_+) \subset \mathcal{D}(Q_+^\dagger)$, since Q_+ is symmetric.

We will now determine the properties of the functions in $\mathcal{D}(Q_+^\dagger)$, and the way Q_+^\dagger acts on them. In a distributional sense, equation (3.4) implies that

$$-\phi_1' + \left(\frac{g}{x} - x\right)\phi_1 = \sqrt{2}\chi_2, \quad (3.5)$$

$$\phi_2' + \left(\frac{g}{x} - x\right)\phi_2 = \sqrt{2}\chi_1, \quad (3.6)$$

which shows that $\Phi'(x)$ is a regular (locally integrable) distribution. This implies that $\Phi(x)$ is an absolutely continuous function for $x > 0$.

Therefore, the domain of Q_+^\dagger consists of those (square-integrable) absolutely continuous functions such that the left-hand sides in equations (3.5) and (3.6) are also square-integrable functions on the half-line:

$$\mathcal{D}(Q_+^\dagger) = \{\Phi \in AC(\mathbb{R}^+ \setminus \{0\}) \cap \mathbf{L}_2(\mathbb{R}^+) : A\phi_1, \tilde{A}\phi_2 \in \mathbf{L}_2(\mathbb{R}^+)\}. \quad (3.7)$$

¹ See theorem X.25 on page 180 of [19].

Consequently, an integration by parts on the left-hand side of equation (3.4) is justified, and we conclude that the action of Q_+^\dagger on $\Phi \in \mathcal{D}(Q_+^\dagger)$ also reduces to the application of the differential operator

$$Q_+^\dagger \Phi = \begin{pmatrix} 0 & \tilde{A} \\ A & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \quad (3.8)$$

3.2. Spectrum of Q_+^\dagger

We now consider the eigenvalue problem for Q_+^\dagger ,

$$Q_+^\dagger \Phi_\lambda = \lambda \Phi_\lambda, \quad (3.9)$$

or equivalently

$$A\phi_1 = \lambda\phi_2, \quad \tilde{A}\phi_2 = \lambda\phi_1, \quad (3.10)$$

with

$$\Phi_\lambda = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in \mathcal{D}(Q_+^\dagger) \quad (3.11)$$

and $\lambda \in \mathbb{C}$.

From equations (2.6), (2.7) and (3.10), it follows immediately that $\Phi_\lambda'(x)$ is an absolutely continuous function. In fact, the successive applications of Q_+^\dagger on both sides of equation (3.9) show that $\Phi_\lambda(x) \in C^\infty(\mathbb{R}^+ \setminus \{0\})$, and equation (3.10) is just a system of ordinary differential equations.

Replacing ϕ_2 in terms of ϕ_1 we get

$$-\frac{1}{2}\phi_1'' + \frac{1}{2} \left\{ \frac{g(g-1)}{x^2} + x^2 - 1 - 2g \right\} \phi_1 = \lambda^2 \phi_1, \quad (3.12)$$

$$\lambda\phi_2 = \frac{1}{\sqrt{2}} \left\{ -\phi_1' + \left(\frac{g}{x} - x \right) \phi_1 \right\}. \quad (3.13)$$

Making the substitution

$$\phi_1(x) = x^g e^{-x^2/2} F(x^2) \quad (3.14)$$

in equation (3.12) we get the Kummer equation [27] for $F(z)$,

$$zF''(z) + (b-z)F'(z) - aF(z) = 0, \quad (3.15)$$

with

$$a = -\frac{\lambda^2}{2}, \quad b = g + \frac{1}{2}. \quad (3.16)$$

For any values of the parameters a and b , equation (3.15) has two linearly independent solutions [27] given by the Kummer function

$$y_1(z) = U(a, b, z) = \frac{\pi}{\sin \pi b} \left\{ \frac{M(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b} \frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)} \right\}, \quad (3.17)$$

and

$$y_2(z) = e^z U(b-a, b, -z). \quad (3.18)$$

In equation (3.17), $M(a, b, z) = {}_1F_1(a; b; z)$ is the confluent hypergeometric function.

Since for large values of the argument [27]

$$U(a, b, z) = z^{-a} \{1 + \mathcal{O}(|z|^{-1})\}, \quad (3.19)$$

only $y_1(x^2)$ leads to a function $\phi_1(x) \in \mathbf{L}_2(1, \infty)$ when replaced in equation (3.14), while $y_2(x^2)$ should be discarded.

Therefore, we get

$$\phi_1(x) = x^g e^{-x^2/2} U\left(-\frac{\lambda^2}{2}, g + \frac{1}{2}, x^2\right). \quad (3.20)$$

On the other hand, replacing equation (3.20) in equation (3.13), it is straightforward to show that [27]

$$\phi_2(x) = -\frac{\lambda}{\sqrt{2}} x^{g+1} e^{-x^2/2} U\left(1 - \frac{\lambda^2}{2}; g + \frac{3}{2}, x^2\right), \quad (3.21)$$

which is also in $\mathbf{L}_2(1, \infty)$.

In order to determine the spectrum of Q_+^\dagger , we must now consider the behaviour of $\Phi_\lambda(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$ near the origin. From equation (3.17), and the small argument expansion of Kummer's functions (see [27], p 508), one can straightforwardly show that three cases should be distinguished, according to the values of the coupling g :

- If $\{g \geq 1/2\}$, it can be seen that $\Phi_\lambda(x) \notin \mathbf{L}_2(0, 1)$ unless $-\lambda^2/2 = -n$, with $n = 0, 1, 2, \dots$. In this case, taking into account that $U(-n, b, z)$ reduces to a Laguerre polynomial (of degree n in z),

$$U(-n, b, z) = (-1)^n n! L_n^{(b-1)}(z), \quad (3.22)$$

we have $\phi_1(x) \sim x^g$ and $\phi_2(x) \sim x^{g+1}$ for $0 < x \ll 1$. (Note that the square-integrability of $\phi_1(x)$ and $\phi_2(x)$ on \mathbb{R}^+ is guaranteed by the decreasing exponentials in equations (3.20) and (3.21).) Therefore, in this region Q_+^\dagger has a symmetric real spectrum given by the (degeneracy one) eigenvalues

$$\lambda_0 = 0, \quad \lambda_{\pm, n} = \pm\sqrt{2n}, \quad n = 1, 2, 3, \dots \quad (3.23)$$

corresponding to the eigenfunctions

$$\Phi_0 = x^g e^{-x^2/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (3.24)$$

and

$$\Phi_{\pm, n} = (-1)^n n! x^g e^{-x^2/2} \begin{pmatrix} L_n^{(g-\frac{1}{2})}(x^2) \\ \mp \frac{x}{\sqrt{n}} L_{n-1}^{(g+\frac{1}{2})}(x^2) \end{pmatrix} \quad (3.25)$$

respectively.

- For $\{-1/2 < g < 1/2\}$, it can be seen from (3.20), (3.21) and (3.17) that $\Phi_\lambda(x) \in \mathbf{L}_2(0, 1), \forall \lambda \in \mathbb{C}$. This means that, for these values of g , every complex number is an eigenvalue of Q_+^\dagger with degeneracy one. For example, the eigenfunction of Q_+^\dagger corresponding to $\lambda = i$ is given by

$$\Phi_{\lambda=i}(x) = \Phi_+(x) = \begin{pmatrix} \phi_{+,1} \\ \phi_{+,2} \end{pmatrix} = x^g e^{-x^2/2} \begin{pmatrix} U\left(\frac{1}{2}, g + \frac{1}{2}, x^2\right) \\ -\frac{i}{\sqrt{2}} x U\left(\frac{3}{2}, g + \frac{3}{2}, x^2\right) \end{pmatrix}, \quad (3.26)$$

while the eigenfunction corresponding to $\lambda = -i$ is given by its complex conjugate,

$$\Phi_{\lambda=-i}(x) = \Phi_-(x) = \Phi_+(x)^* \quad (3.27)$$

(since the coefficients in the differential operators in equation (3.10) are real).

- Finally, for $g \leq -1/2$, it can be seen that $\Phi_\lambda(x) \notin \mathbf{L}_2(0, 1)$ unless $-\lambda^2/2 = g - \frac{1}{2} - n$, with $n = 0, 1, 2, \dots$. In this case, taking into account the Kummer transformation (see [27], p 505),

$$U(1 - n - b, 2 - b, z) = z^{b-1}U(-n, b, z), \tag{3.28}$$

and equation (3.22), we have $\phi_1(x) \sim x^{1-g}$ and $\phi_2(x) \sim x^{-g}$ for $0 < x \ll 1$. Therefore, in this region Q_+^\dagger has a symmetric real spectrum given by the (degeneracy one) eigenvalues

$$\lambda_{\pm,n} = \pm\sqrt{2n + 1 - 2g}, \quad n = 0, 1, 2, \dots \tag{3.29}$$

corresponding to the eigenfunctions

$$\Phi_{\pm,n} = (-1)^n n! \times x^{-g} e^{-x^2/2} \begin{pmatrix} x L_n^{(\frac{1}{2}-g)}(x^2) \\ \mp \sqrt{n + \frac{1}{2} - g} L_n^{(-g-\frac{1}{2})}(x^2) \end{pmatrix}. \tag{3.30}$$

Note that no eigenvalue vanishes for these values of the coupling.

These results will be employed in the next section to determine the self-adjoint extensions of Q_+ .

4. Self-adjoint extensions of Q_+

According to von Neumann’s theory [19], to construct the self-adjoint extensions of Q_+ we must take into account the different behaviours of Q_+^\dagger , described in the previous section.

4.1. For $|g| \geq 1/2$ the operator Q_+ is essentially self-adjoint

As seen in section 3.2, the *deficiency indices* [19] of Q_+ , defined as the dimensions of the deficiency subspaces \mathcal{K}_\pm ,

$$n_\pm := \dim \text{Ker}(Q_+^\dagger \mp i), \tag{4.1}$$

vanish for $|g| \geq 1/2$. This means that Q_+ is *essentially self-adjoint* [19] in these regions of the coupling, admitting there a unique self-adjoint extension given by Q_+^\dagger (which, in this case, is itself a self-adjoint operator).

The corresponding self-adjoint extension of the Hamiltonian in equation (2.10) is given by (see footnote 1)

$$\overline{H} = (Q_+^\dagger)^2, \tag{4.2}$$

where the operator composition on the right-hand side is possible in the dense domain

$$\mathcal{D}(\overline{H}) = \{\psi \in \mathcal{D}(Q_+^\dagger) : Q_+^\dagger \psi \in \mathcal{D}(Q_+^\dagger)\}. \tag{4.3}$$

Note that every eigenfunction of Q_+^\dagger , corresponding to an eigenvalue λ , belongs to $\mathcal{D}(\overline{H})$. Therefore, it is also an eigenfunction of \overline{H} with eigenvalue $E = \lambda^2$. So, we have:

- For $g \geq 1/2$, the eigenfunctions of \overline{H} are given in equations (3.24) and (3.25). Note that there is a unique zero mode, while the positive eigenvalues of \overline{H} ,

$$E_n = 2n, \quad n = 1, 2, 3, \dots \tag{4.4}$$

are doubly degenerate (see equation (3.23)). One can add and subtract the corresponding eigenfunctions in equation (3.25) to get *bosonic* and *fermionic* states (with only the upper and lower components non-vanishing respectively). For these values of the coupling, the Witten index is $\Delta = 1$ and the SUSY is manifest [8].

- For $g \leq -1/2$, the eigenfunctions of \overline{H} are given in equation (3.30). Note that there is no zero mode. Once again, the positive eigenvalues of \overline{H} ,

$$E_n = 2n + 1 - 2g \geq 2, \quad n = 0, 1, 2, \dots \tag{4.5}$$

are doubly degenerate (see equation (3.29)), and the eigenfunctions can be combined to get bosonic and fermionic states. For these values of g , the SUSY is spontaneously broken and the Witten index is $\Delta = 0$ [8].

4.2. For $|g| < 1/2$ the operator Q_+ is not essentially self-adjoint

On the other hand, according to equations (3.26) and (3.27) in section 3.2, for $-1/2 < g < 1/2$ the deficiency indices are $n_{\pm} = 1$. In this region Q_+ admits a one-parameter family of self-adjoint extensions, Q_+^γ , which are in a one-to-one correspondence with the isometries from \mathcal{K}_+ onto \mathcal{K}_- [19], characterized by

$$U(\gamma)\Phi_+(x) := e^{2i\gamma}\Phi_-, \quad \gamma \in [0, \pi), \tag{4.6}$$

with Φ_+ and Φ_- given in equations (3.26) and (3.27) respectively.

The self-adjoint operator Q_+^γ is the restriction of Q_+^\dagger to a dense subspace

$$\mathcal{D}(Q_+^\gamma) \subset \mathcal{D}(Q_+^\dagger) = \mathcal{D}(\overline{Q}_+) \oplus \mathcal{K}_+ \oplus \mathcal{K}_- \tag{4.7}$$

(here \overline{Q}_+ is the closure of Q_+ [19]), composed by those functions which can be written as

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \overline{\Phi}_0 + c(\Phi_+ + e^{2i\gamma}\Phi_-), \tag{4.8}$$

with $\overline{\Phi}_0 = \begin{pmatrix} \phi_{0,1} \\ \phi_{0,2} \end{pmatrix} \in \mathcal{D}(\overline{Q}_+)$, and the constant $c \in \mathbb{C}$.

Obviously, we have

$$Q_+^\gamma \Phi = Q_+^\dagger \overline{\Phi}_0 + ic(\Phi_+ - e^{2i\gamma}\Phi_-), \tag{4.9}$$

with Q_+^\dagger given in equation (3.8).

Equation (4.8) completely characterizes the behaviour near the origin of the functions $\Phi \in \mathcal{D}(Q_+^\gamma)$. As we will see, it also allows us to determine the spectrum of Q_+^γ .

Indeed, in appendix A we have worked out the domain of the closure of Q_+ , $\mathcal{D}(\overline{Q}_+)$, showing that

$$\phi_{0,1}(x) = o(x^g), \quad \phi_{0,2}(x) = o(x^{-g}), \tag{4.10}$$

for $x \rightarrow 0^+$. On the other side, from equations (3.11), (3.20), (3.21) and (3.17), one can easily see that the components of any eigenfunction Φ_λ of Q_+^\dagger behave as

$$\phi_1(x) = \frac{\Gamma(\frac{1}{2} - g)}{\Gamma(\frac{1-\lambda^2}{2} - g)} x^g + O(x^{1-g}), \quad \phi_2(x) = \frac{\sqrt{2}}{\lambda} \frac{\Gamma(\frac{1}{2} + g)}{\Gamma(-\frac{\lambda^2}{2})} x^{-g} + O(x^{1+g}). \tag{4.11}$$

Therefore, no eigenfunction of Q_+^\dagger belongs to $\mathcal{D}(\overline{Q}_+)$.

Consequently, it is the contributions of Φ_{\pm} in equation (4.8) which determine the spectrum of Q_+^γ . In fact, consider the limit

$$\lim_{x \rightarrow 0^+} \frac{x^{-g}\phi_1(x)}{x^g\phi_2(x)} = \frac{\lambda}{\sqrt{2}} \frac{\Gamma(-\frac{\lambda^2}{2})}{\Gamma(\frac{1-\lambda^2}{2} - g)} \frac{\Gamma(\frac{1}{2} - g)}{\Gamma(\frac{1}{2} + g)}. \tag{4.12}$$

For a non-vanishing c on the right-hand side of equation (4.8), this limit must coincide with

$$\lim_{x \rightarrow 0^+} \frac{\text{Re}\{e^{-i\gamma}x^{-g}\phi_{+,1}(x)\}}{\text{Re}\{e^{-i\gamma}x^g\phi_{+,2}(x)\}} = -\sqrt{\frac{\pi}{2}} \frac{\cot(\gamma)}{\Gamma(1-g)} \frac{\Gamma(\frac{1}{2} - g)}{\Gamma(\frac{1}{2} + g)}, \tag{4.13}$$

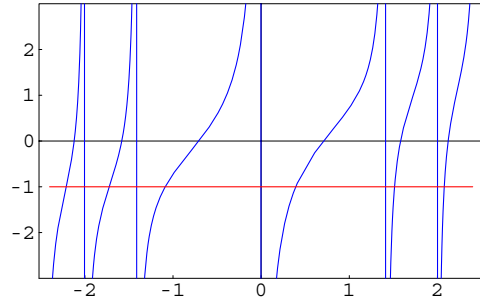


Figure 1. $f(\lambda) := \frac{\lambda\Gamma(-\frac{\lambda^2}{2})}{\Gamma(\frac{1-\lambda^2}{2}-g)}$ for $g = 1/4$, and $\beta(\gamma) \equiv -1$.

where equations (3.27) and (4.11) with $\lambda \rightarrow i$ have been taken into account. Then, the eigenvalues of Q_+^γ (which are real) are the solutions of the transcendental equation

$$f(\lambda) := \frac{\lambda\Gamma(-\frac{\lambda^2}{2})}{\Gamma(\frac{1-\lambda^2}{2}-g)} = -\frac{\sqrt{\pi} \cot(\gamma)}{\Gamma(1-g)} =: \beta(\gamma). \tag{4.14}$$

Note that $f(\lambda)$ is an odd function of λ , and $-\infty \leq \beta(\gamma) < \infty$ for $0 \leq \gamma < \pi$.

The function $f(\lambda)$ on the left-hand side of equation (4.14) has been plotted in figure 1, for a value of the coupling $g = 1/4$. The eigenvalues of Q_+^γ are determined by the intersections of the graphic of $f(\lambda)$ with the horizontal line corresponding to the constant $\beta(\gamma)$ (taken equal to -1 in the figure). As stressed in section 3.2, the eigenvalues are non-degenerate. The eigenfunctions are obtained by replacing these eigenvalues in equations (3.11), (3.20) and (3.21).

It can easily be seen that, in general, the spectrum is non-symmetric with respect to the origin. The exceptions are the self-adjoint extensions corresponding to $\gamma = 0$ ($\beta = -\infty$) and $\gamma = \pi/2$ ($\beta = 0$). Indeed, the condition $f(-\lambda) = f(\lambda)$ for a non-vanishing λ requires that

$$\frac{1}{\Gamma(-\frac{\lambda^2}{2})\Gamma(\frac{1-\lambda^2}{2}-g)} = 0, \tag{4.15}$$

whose solutions (see figure 1) correspond to the intersections with the constant $\beta = -\infty$,

$$-\frac{\lambda^2}{2} = -n \Rightarrow \lambda_{\pm,n} = \pm\sqrt{2n}, \quad n = 1, 2, 3, \dots \tag{4.16}$$

or the constant $\beta = 0$,

$$\frac{1-\lambda^2}{2} - g = -n \Rightarrow \lambda_{\pm,n} = \pm\sqrt{2n+1-2g}, \quad n = 0, 1, 2, \dots \tag{4.17}$$

In particular, $Q_+^{\gamma=0}$ is the only self-adjoint extension having a zero mode. For $0 < \gamma < \pi$, the eigenvalues are contained between contiguous asymptotes of $\Gamma(-\frac{\lambda^2}{2})$,

$$\sqrt{2n} < |\lambda_{\pm,n}| < \sqrt{2(n+1)}. \tag{4.18}$$

Now, for a given Q_+^γ , with $\gamma \in [0, \pi)$, we get the self-adjoint extension of the Hamiltonian defined by (see footnote 1)

$$H_\gamma = (Q_+^\gamma)^2 \equiv (Q_+^\dagger)^2|_{\mathcal{D}(H_\gamma)}, \tag{4.19}$$

where the operator composition on the right-hand side is the restriction of $(Q_+^\dagger)^2$ to the dense subspace

$$\mathcal{D}(H_\gamma) = \{\psi \in \mathcal{D}(Q_+^\gamma) : Q_+^\dagger \psi \in \mathcal{D}(Q_+^\gamma)\}. \quad (4.20)$$

This domain includes, in particular, all the eigenfunctions of Q_+^γ , which are then also eigenvectors of H_γ :

$$Q_+^\gamma \Phi_\lambda = \lambda \Phi_\lambda \Rightarrow H_\gamma \Phi_\lambda = \lambda^2 \Phi_\lambda. \quad (4.21)$$

Note that, except for the special values $\gamma = 0, \pi/2$, the spectrum of H_γ is non-degenerate.

Three cases can be distinguished:

- For $\gamma = 0$ ($\beta = -\infty$) we get the only self-adjoint extension of H having a (non-degenerate) zero mode. The corresponding eigenfunction is also given by equation (3.24). From equation (4.16), it follows that the non-vanishing eigenvalues of $H_{\gamma=0}$ are doubly degenerate,

$$E_{\pm,n} = (\lambda_{\pm,n})^2 = 2n, \quad n = 1, 2, 3, \dots \quad (4.22)$$

We can take linear combinations of the corresponding eigenfunctions, $\Phi_{\pm,n}$ (given by equation (3.25), with $|g| < 1/2$), to get linearly independent states with only one non-vanishing component.

Therefore, the conditions imposed on the functions in $\mathcal{D}(Q_+^{\gamma=0})$ by equation (4.8) with $\gamma = 0$ give rise to a manifestly supersymmetric self-adjoint extension of the Hamiltonian H . The Witten index is in this case $\Delta = 1$.

- For $\gamma = \pi/2$ ($\beta = 0$) we get a self-adjoint extension of H with no zero modes, and a doubly degenerate spectrum. Indeed, from equation (4.17) it follows that the self-energies of $H_{\gamma=\pi/2}$ are

$$E_{\pm,n} = (\lambda_{\pm,n})^2 = 2n + 1 - 2g, \quad n = 0, 1, 2, \dots \quad (4.23)$$

These eigenvalues are positive, since $1 - 2g > 0$. The eigenfunctions $\Phi_{\pm,n}$, whose expressions are given by equation (3.30) with $|g| < 1/2$, can be combined to get bosonic and fermionic states.

In the present case, the conditions imposed on the functions in $\mathcal{D}(Q_+^{\gamma=\pi/2})$ by equation (4.8) with $\gamma = \pi/2$ break the SUSY, preserving the degeneracy of the spectrum. This gives a Witten index $\Delta = 0$.

- For $\gamma \neq 0, \pi/2$ we get self-adjoint extensions of H with no zero modes and non-degenerate spectra. The eigenvalues of H_γ (the square of those λ solutions of equation (4.14)) are all positive, and the corresponding eigenfunctions are neither bosonic nor fermionic states. (See equations (3.20) and (3.21).)

In this case, the condition imposed in equation (4.8) to select the domain of Q_+^γ breaks not only the SUSY, but also the degeneracy of the spectrum. The Witten index is $\Delta = 0$.

The analysis performed in this section should be compared with the results obtained in [20], where even and odd solutions for a regularized version of this superpotential are worked out, obtaining in the limit eigenfunctions belonging to the domains of two different self-adjoint Hamiltonians, those corresponding to $\gamma = 0$ and $\gamma = \pi/2$.

4.3. The $g = 0$ case

It is instructive to consider the $g = 0$ case, in which the superpotential (equation (2.5)) is regular at the origin, and the functions in $\mathcal{D}(Q_+^\gamma)$ approach constants for $x \rightarrow 0^+$.

Indeed, if $g = 0$, we have for the functions on the right-hand side of equation (4.8) (see equations (A.6) and (4.11))

$$\bar{\Phi}_0(x) = o(x^0), \quad \Phi_+(x) + e^{2i\gamma} \Phi_-(x) = 2^{3/2} e^{i\gamma} \begin{pmatrix} \sqrt{\frac{\pi}{2}} \cos \gamma \\ -\sin \gamma \end{pmatrix} + O(x). \quad (4.24)$$

Therefore, the domain of Q_+^γ can be characterized simply by a local boundary condition of the form

$$\Phi \in \mathcal{D}(Q_+^\gamma) \Rightarrow \begin{pmatrix} \sin \gamma & \sqrt{\frac{\pi}{2}} \cos \gamma \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} = 0. \quad (4.25)$$

The particular values $\gamma = 0$ and $\gamma = \pi/2$ imply to demand $\phi_2(0) = 0$ and $\phi_1(0) = 0$, respectively.

As discussed in section 4.2, for $\gamma = 0$ the SUSY is manifest: there is a zero mode of $H_{\gamma=0}$,

$$\Phi_0 = \begin{pmatrix} e^{-x^2/2} \\ 0 \end{pmatrix}, \quad (4.26)$$

and the eigenfunctions corresponding to the (doubly degenerate) non-vanishing eigenvalues, $E_{\pm,n} = 2n, n = 1, 2, \dots$, reduce to (see equations (4.22) and (3.25))

$$\Phi_{\pm,n}(x) = \frac{e^{-x^2/2}}{2^{2n}} \begin{pmatrix} H_{2n}(x) \\ \pm 2\sqrt{n} H_{2n-1}(x) \end{pmatrix}, \quad (4.27)$$

where $H_n(x)$ are the Hermite polynomials. Note that the lower component and the first derivative of the upper component of the eigenvectors vanish at the origin.

For $\gamma = \pi/2$, the SUSY is spontaneously broken: there are no zero modes, and the eigenfunctions of $H_{\gamma=\pi/2}$ corresponding to the (doubly degenerate) non-vanishing eigenvalues, $E_{\pm,n} = 2n + 1, n = 0, 1, \dots$, reduce to (see equations (4.23) and (3.30))

$$\Phi_{\pm,n}(x) = \frac{e^{-x^2/2}}{2^{2n+1}} \begin{pmatrix} H_{2n+1}(x) \\ \mp \sqrt{4n + 2} H_{2n}(x) \end{pmatrix}. \quad (4.28)$$

In this case, the upper component and the first derivative of the lower component of the eigenvectors vanish at the origin.

For other values of the parameter γ , the SUSY is also broken: there are no zero modes and the spectrum is non-degenerate, as previously discussed.

Therefore, we see that all except one of the possible local boundary conditions at the origin defining a self-adjoint supercharge Q_+^γ (and a self-adjoint Hamiltonian H_γ), equation (4.25), break the SUSY.

This should be compared with the results obtained in [20] for the super half-oscillator, where the regularization employed for the superpotential automatically leads to the eigenvalues and eigenfunctions corresponding to the $\gamma = 0$ case, equations (4.26) and (4.27), for which SUSY is manifest.

5. Discussion

In the previous sections, we have seen how to choose suitable domains to define self-adjoint extensions of the supercharge Q_+ , initially defined in the restricted domain $C_0^\infty(\mathbf{R}^+ \setminus \{0\})$ as in equations (2.8), (2.6) and (2.7).

As stressed in section 2, Q_+ and Q_- are related by a unitary transformation (see equation (2.12)). Then, each self-adjoint extension of the first, Q_+^γ , determines a self-adjoint extension of the second, Q_-^γ , whose domain is obtained from $\mathcal{D}(Q_+^\gamma)$ through the unitary transformation $e^{i\pi\sigma_3/4}$,

$$\mathcal{D}(Q_-^\gamma) = \{\Psi : e^{-i\pi\sigma_3/4}\Psi \in \mathcal{D}(Q_+^\gamma)\} = e^{i\pi\sigma_3/4}(\mathcal{D}(Q_+^\gamma)). \quad (5.1)$$

Consequently, Q_-^γ is an equivalent representation of the self-adjoint supercharge Q_+^γ , sharing both operators the same spectrum.

Similarly, its square $(Q_-^\gamma)^2$, defined on the dense subspace (see footnote 1)

$$\mathcal{D}((Q_-^\gamma)^2) = \{\Psi \in \mathcal{D}(Q_-^\gamma) : Q_-^\gamma\Psi \in \mathcal{D}(Q_-^\gamma)\} = e^{i\pi\sigma_3/4}(\mathcal{D}(H^\gamma)), \quad (5.2)$$

is an equivalent representation of the self-adjoint extension $H^\gamma = (Q_+^\gamma)^2$ of the Hamiltonian H , initially defined on $C_0^\infty(\mathbb{R}^+ \setminus \{0\})$ as in equation (2.10).

These equivalent representations of the Hamiltonian coincide only if the domain $\mathcal{D}(Q_+^\gamma)$ is left invariant by the unitary transformation $e^{i\pi\sigma_3/4}$, and this occurs only for the particular self-adjoint extensions corresponding to $\gamma = 0$ and $\gamma = \pi/2$ (extensions for which states can be chosen to be *bosons* or *fermions*), as can easily be seen from equation (4.13).

Consequently, the operator compositions

$$(Q_+^\gamma)^2, \quad (Q_-^\gamma)^2, \quad Q_+^\gamma Q_-^\gamma \quad \text{and} \quad Q_-^\gamma Q_+^\gamma \quad (5.3)$$

make sense in the same (dense) domain $\mathcal{D}(H^\gamma)$ only for $\gamma = 0, \pi/2$, values of the parameter characterizing self-adjoint extensions for which the $N = 2$ SUSY algebra is realized,

$$\{Q_+^\gamma, Q_-^\gamma\} = 0, \quad H^\gamma = (Q_+^\gamma)^2 = (Q_-^\gamma)^2. \quad (5.4)$$

For other values of the parameter γ , $\mathcal{D}(Q_+^\gamma)$ is not left invariant by $e^{i\pi\sigma_3/4}$, and there is no dense domain in the Hilbert space where the self-adjoint operator compositions in equation (5.3) could be defined.

Therefore, for $\gamma \neq 0, \pi/2$ only one self-adjoint supercharge can be defined in the domain of the Hamiltonian, and the SUSY algebra reduces to the $N = 1$ case,

$$H^\gamma = (Q_+^\gamma)^2 \quad (5.5)$$

(or, equivalently, $(Q_-^\gamma)^2$).

At this point, it is worthwhile remarking that the double degeneracy of the non-vanishing eigenvalues of H^γ with $\gamma = 0, \pi/2$ is a consequence of the existence of a second supercharge. Indeed, if

$$Q_+^\gamma\Phi_\lambda = \lambda\Phi_\lambda, \quad (5.6)$$

with $\Phi_\lambda \in \mathcal{D}(H^\gamma)$ and $\lambda \neq 0$, then equations (5.4) imply that

$$Q_+^\gamma(Q_-^\gamma\Phi_\lambda) = -Q_-^\gamma(Q_+^\gamma\Phi_\lambda) = -\lambda(Q_-^\gamma\Phi_\lambda). \quad (5.7)$$

Then, $Q_-^\gamma\Phi_\lambda$ ($\in \mathcal{D}(Q_-^\gamma) \equiv \mathcal{D}(Q_+^\gamma)$) is a linearly independent eigenvector of Q_+^γ corresponding to the eigenvalue $-\lambda$, since $Q_-^\gamma\Phi_\lambda \perp \Phi_\lambda$ and

$$\|Q_-^\gamma\Phi_\lambda\|^2 = (\Phi_\lambda, (Q_-^\gamma)^2\Phi_\lambda) = (\Phi_\lambda, H^\gamma\Phi_\lambda) = \lambda^2\|\Phi_\lambda\|^2 \neq 0. \quad (5.8)$$

In conclusion, we see that for a general self-adjoint extension of the supercharge Q_+^γ (and the corresponding extension of the Hamiltonian, H^γ), the conditions the functions contained in $\mathcal{D}(H^\gamma)$ satisfy near the origin prevent the $N = 2$ SUSY, losing one supercharge. Then, only the $N = 1$ SUSY algebra is realized, giving rise to a non-symmetric (and non-degenerate) spectrum for the remaining supercharge, and a non-degenerate spectrum for the Hamiltonian. The remaining $N = 1$ SUSY is spontaneously broken since there are no zero modes.

The only exceptions are those self-adjoint extensions corresponding to $\gamma = 0$ and $\gamma = \pi/2$, for which the $N = 2$ SUSY algebra can be realized. In these two cases the supercharges have a common symmetric (non-degenerate) spectrum and the excited states of the Hamiltonian are doubly degenerate.

For $\gamma = 0$, the (non-degenerate) ground state of H^0 has a vanishing energy and the SUSY is manifest, while for $\gamma = \pi/2$ the (doubly degenerate) ground state of $H^{\pi/2}$ has positive energy and the SUSY is spontaneously broken.

It is also worthwhile pointing out that $N = 2$ SUSY can be realized only when the supercharge domain $\mathcal{D}(Q_+^\gamma)$ is scale invariant. Indeed, consider a function $\Phi(x) \in \mathcal{D}(Q_+^\gamma)$; under the scaling isometry

$$T_a \Phi(x) := a^{1/2} \Phi(ax), \tag{5.9}$$

with $a > 0$, the limit on the left-hand side of equation (4.12) becomes

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{x^{-g} (T_a \Phi)_1(x)}{x^g (T_a \Phi)_2(x)} &= \lim_{x \rightarrow 0^+} \frac{x^{-g} a^{1/2} \phi_1(ax)}{x^g a^{1/2} \phi_2(ax)} \\ &= a^{2g} \lim_{y \rightarrow 0^+} \frac{y^{-g} \phi_1(y)}{y^g \phi_2(y)} = -\sqrt{\frac{\pi}{2}} \frac{a^{2g} \cot(\gamma) \Gamma(\frac{1}{2} - g)}{\Gamma(1 - g) \Gamma(\frac{1}{2} + g)}, \end{aligned} \tag{5.10}$$

where equation (4.13) has been used in the last step. This shows that $T_a \Phi(x)$ belongs to the domain of the self-adjoint extension $Q_+^{\gamma_a}$ characterized by the parameter γ_a satisfying

$$\cot(\gamma_a) = a^{2g} \cot(\gamma). \tag{5.11}$$

Obviously, $\gamma_a = \gamma, \forall a > 0$, only for $\gamma = 0, \pi/2$. For other values of γ the conditions the functions in $\mathcal{D}(Q_+^\gamma)$ satisfy near the origin are not scale invariant.

Finally let us stress that, as remarked in the introduction, when the formal expression of the Hamiltonian as a differential operator is not essentially self-adjoint, additional information is needed to identify the self-adjoint extension which correctly describes the properties of the physical system.

For the particular case under consideration we have seen that, even though we have started from the *formal* $N = 2$ SUSY algebra of equations (2.1), (2.2), (2.3), (2.6) and (2.7), we find a whole family of self-adjoint extensions offering the possibility of having not only spontaneously broken SUSY, but also a non-degenerate Hamiltonian spectrum.

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Appendix A. Closure of Q_+

In this section we will justify to disregard the contributions of the functions in $\mathcal{D}(\overline{Q}_+)$ to the $x \rightarrow 0^+$ limit of the right-hand side of equation (4.8). In fact, we will show that, near the origin, $\overline{\Phi}_0(x) \in \mathcal{D}(\overline{Q}_+)$ behaves as in equation (4.10), for every $|g| < 1/2$.

Since the graph of Q_+ is contained in the graph of Q_+^\dagger , which is a closed set [19], it is sufficient to determine the closure of the former. In doing so, we must consider those Cauchy sequences

$$\left\{ \Psi_n = \begin{pmatrix} \psi_{1,n} \\ \psi_{2,n} \end{pmatrix} \right\}_{n \in \mathbb{N}} \subset \mathcal{D}(Q_+) := C_0^\infty(\mathbb{R}^+ \setminus \{0\}) \tag{A.1}$$

such that $\{Q_+ \Psi_n\}_{n \in \mathbb{N}}$ are also Cauchy sequences.

In this case, in particular, $\{\psi_{1,n}\}_{n \in \mathbb{N}}$, $\{\psi_{2,n}\}_{n \in \mathbb{N}}$, $\{A\psi_{1,n}\}_{n \in \mathbb{N}}$ and $\{\tilde{A}\psi_{2,n}\}_{n \in \mathbb{N}}$ are Cauchy sequences in $\mathbf{L}_2(0, 1)$, with A and \tilde{A} given in equations (2.6) and (2.7) respectively.

Moreover, since x is bounded in $[0, 1]$, and the sum of fundamental sequences is also fundamental, it follows that $\{\psi'_{1,n}(x) - \frac{g}{x}\psi_{1,n}(x)\}_{n \in \mathbb{N}}$, and $\{\psi'_{2,n}(x) + \frac{g}{x}\psi_{2,n}(x)\}_{n \in \mathbb{N}}$ are Cauchy sequences in $\mathbf{L}_2(0, 1)$.

On the other hand, we have $x^{\pm g} \in \mathbf{L}_2(0, 1)$ for any $-1/2 < g < 1/2$. Therefore,

$$\left\{x^{-g} \left(\psi'_{1,n}(x) - \frac{g}{x} \psi_{1,n}(x) \right)\right\}_{n \in \mathbb{N}} = \{(x^{-g} \psi_{1,n}(x))'\}_{n \in \mathbb{N}} \quad (\text{A.2})$$

and

$$\left\{x^g \left(\psi'_{2,n}(x) + \frac{g}{x} \psi_{2,n}(x) \right)\right\}_{n \in \mathbb{N}} = \{(x^g \psi_{2,n}(x))'\}_{n \in \mathbb{N}} \quad (\text{A.3})$$

are Cauchy sequences in $\mathbf{L}_1(0, 1)$.

Now, taking into account that these functions vanish identically in a neighbourhood of the origin, one can see that $\{x^{-g}\psi_{1,n}(x)\}_{n \in \mathbb{N}}$ and $\{x^g\psi_{2,n}(x)\}_{n \in \mathbb{N}}$ converge uniformly in $[0, 1]$. Indeed, $\forall x \in [0, 1]$ we have

$$\begin{aligned} |x^{-g}[\psi_{1,n}(x) - \psi_{1,m}(x)]| &= \left| \int_0^x (y^{-g}[\psi_{1,n}(y) - \psi_{1,m}(y)])' dy \right| \\ &\leq \|(y^{-g}\psi_{1,n}(y))' - (y^{-g}\psi_{1,m}(y))'\|_{\mathbf{L}_1(0,1)} \rightarrow_{n,m \rightarrow \infty} 0, \end{aligned} \quad (\text{A.4})$$

and similarly for the second sequence.

Consequently, there are two continuous functions, $x^{-g}\phi_{0,1}(x)$ and $x^g\phi_{0,2}(x)$, which are the *uniform* limits in $[0, 1]$

$$x^{-g}\phi_{0,1}(x) = \lim_{n \rightarrow \infty} x^{-g}\psi_{1,n}(x), \quad x^g\phi_{0,2}(x) = \lim_{n \rightarrow \infty} x^g\psi_{2,n}(x). \quad (\text{A.5})$$

In particular, we get

$$\lim_{x \rightarrow 0} x^{-g}\phi_{0,1}(x) = 0, \quad \lim_{x \rightarrow 0} x^g\phi_{0,2}(x) = 0. \quad (\text{A.6})$$

Moreover, the limit of the sequence in $\mathbf{L}_2(0, 1)$ is given by

$$\lim_{n \rightarrow \infty} \Psi_n = \bar{\Phi}_0 = \begin{pmatrix} \phi_{0,1} \\ \phi_{0,2} \end{pmatrix}. \quad (\text{A.7})$$

Indeed, taking into account that, for any $\varepsilon > 0$,

$$|x^{-g}[\psi_{1,n}(x) - \phi_{0,1}(x)]| < \varepsilon, \quad \forall x \in [0, 1], \quad (\text{A.8})$$

if n is sufficiently large, it follows that

$$\|\psi_{1,n} - \phi_{0,1}\|_{\mathbf{L}_2(0,1)}^2 = \int_0^1 x^{2g} |x^{-g}(\psi_{1,n}(x) - \phi_{0,1}(x))|^2 < \varepsilon^2 \|x^g\|_{\mathbf{L}_2(0,1)}^2, \quad (\text{A.9})$$

and similarly for the lower component.

Equations (A.7) and (A.6) prove our assertion in equation (4.10).

We will finally verify that the so-obtained function $\bar{\Phi}_0$ belongs to $\mathcal{D}(\mathcal{Q}_+^\dagger)$. Let $\rho_1(x)$ be the limit in $\mathbf{L}_1(0, 1)$ of the fundamental sequence given in equation (A.2),

$$\rho_1(x) = \lim_{n \rightarrow \infty} (x^{-g}\psi_{1,n}(x))'. \quad (\text{A.10})$$

Then, given $\varepsilon > 0$, and $\forall x \in [0, 1]$, we have

$$\begin{aligned} \left| x^{-g}\psi_{1,n}(x) - \int_0^x \rho_1(y) dy \right| &= \left| \int_0^x [(y^{-g}\psi_{1,n}(y))' - \rho_1(y)] dy \right| \\ &\leq \|(y^{-g}\psi_{1,n}(y))' - \rho_1(y)\|_{\mathbf{L}_1(0,1)} < \varepsilon, \end{aligned} \quad (\text{A.11})$$

if n is large enough.

Since the uniform limit is unique, it follows from equations (A.5) and (A.11) that

$$\phi_{0,1}(x) = x^g \int_0^x \rho_1(y) dy, \tag{A.12}$$

with $\rho_1 \in L_1(0, 1)$. Therefore, $\phi_{0,1}(x)$ is an absolutely continuous function for $x > 0$. A similar conclusion is obtained for the lower component of $\overline{\Phi}_0$.

Appendix B. Spectral functions associated with Q_+^γ

B.1. The graded partition function

We will now consider the *graded partition function* [28–30] of H^γ , defined as

$$Z_\gamma^F(T) := \text{Tr}\{(-1)^F e^{-TH^\gamma}\} = \sum_{\lambda_n} e^{-T\lambda_n^2} \frac{(\Phi_n, (-1)^F \Phi_n)}{\|\Phi_n\|^2}. \tag{B.1}$$

Subtracting the contribution of a possible zero mode we can write

$$\widehat{Z}_\gamma^F(T) := \sum_{\lambda_n \neq 0} \frac{e^{-T\lambda_n^2}}{\lambda_n} \frac{(Q_+^\dagger \Phi_n, (-1)^F \Phi_n)}{\|\Phi_n\|^2}, \tag{B.2}$$

where

$$(-1)^F \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ -\phi_2 \end{pmatrix}. \tag{B.3}$$

Taking into account equation (3.8), and the fact that the eigenfunctions are real, it is straightforward to get

$$\begin{aligned} \widehat{Z}_\gamma^F(T) &= - \sum_{\lambda_n \neq 0} \frac{e^{-T\lambda_n^2}}{\sqrt{2}\lambda_n \|\Phi_n\|^2} [\phi_{n,1}(x)\phi_{n,2}(x)]_{x=0^+} \\ &= \frac{1}{2} \sum_{\lambda_n \neq 0} \frac{\Gamma(\frac{1}{2} + g)\Gamma(\frac{1}{2} - g) e^{-T\lambda_n^2}}{\Gamma(1 - \frac{\lambda_n^2}{2})\Gamma(\frac{1-\lambda_n^2}{2} - g) \|\Phi_n\|^2}, \end{aligned} \tag{B.4}$$

where the behaviour of the functions in $\mathcal{D}(H^\gamma)$ near the origin (see equation (4.11)) has been taken into account in the last step.

We see that $Z_\gamma^F(T)$ depends on γ though the spectrum of Q_+^γ and, in general, also depends on T . But it can be shown that $Z_\gamma^F(T)$ is independent of T , and coincides with the Witten index, for the particular values $\gamma = 0, \pi/2$.

Indeed, for the eigenvalues of $Q_+^{\gamma=\pi/2}$, given in equation (4.23), each term in the series on the right-hand side of equation (B.4) vanishes because of the second Γ -function in the denominator. So, since there are no zero modes, we get

$$Z_{\gamma=\pi/2}^F(T) \equiv 0 = \Delta_{\gamma=\pi/2}. \tag{B.5}$$

On the other hand, for the eigenvalues of $Q_+^{\gamma=0}$ given in equation (4.22), every term in the series in equation (B.4) vanishes because of the first Γ -function in the denominator. In this case, we get from the zero mode in equation (3.24)

$$Z_{\gamma=0}^F(T) = \frac{(\Phi_0, (-1)^F \Phi_0)}{\|\Phi_0\|^2} = 1 = \Delta_{\gamma=0}. \tag{B.6}$$

For other values of γ , $Z_\gamma^F(T)$ vanishes exponentially with T (since there are no zero modes), reproducing the Witten index in the $T \rightarrow \infty$ limit.

B.2. The spectral asymmetry

The spectrum behaviour for a general self-adjoint extension Q_+^γ , as shown in figure 1, can be characterized by the *spectral asymmetry* [31]

$$\eta(s) := \sum_{\lambda_{\pm,n} \neq 0} \text{sign}(\lambda_{\pm,n}) |\lambda_{\pm,n}|^{-s}. \tag{B.7}$$

Since $|\lambda_{\pm,n}| \sim \sqrt{n}$ (see equation (4.18)), equation (B.7) defines an analytic function on the open half-plane $\text{Re}(s) > 2$.

For the particular values $\beta = -\infty$ and $\beta = 0$, it is evident from equations (4.16) and (4.17) that $\eta(s)$ identically vanishes for any $g \in (-1/2, 1/2)$.

The spectral asymmetry can also be expressed as

$$\eta(s) = \zeta_+(s, \beta) - e^{i\pi s} \zeta_-(s, \beta), \tag{B.8}$$

where

$$\zeta_+(s, \beta) := \sum_{\lambda_{+,n} > 0} \lambda_{+,n}^{-s}, \quad \zeta_-(s, \beta) := \sum_{\lambda_{-,n} < 0} \lambda_{-,n}^{-s}. \tag{B.9}$$

From equation (4.14), it can be seen that (for finite $\beta(\gamma)$) the eigenvalues of Q_+^γ are the zeros of the analytic entire function

$$F(\lambda, \beta) := \frac{\lambda}{\Gamma(\alpha - \frac{\lambda^2}{2})} - \frac{\beta}{\Gamma(-\frac{\lambda^2}{2})}, \tag{B.10}$$

where $\alpha = \frac{1}{2} - g$. Since these zeros are real and simple, we have the following integral representation:

$$\begin{aligned} \zeta_+(s, \beta) &= \frac{1}{2\pi i} \oint_{C_+} \lambda^{-s} \frac{F'(\lambda, \beta)}{F(\lambda, \beta)} d\lambda \\ &= -\frac{1}{2\pi} e^{i\pi s/2} \int_{-\infty+i0}^{\infty+i0} \mu^{-s} \frac{F'(e^{-i\pi/2}\mu, \beta)}{F(e^{-i\pi/2}\mu, \beta)} d\mu, \end{aligned} \tag{B.11}$$

where C_+ encloses counterclockwise the positive zeros of Q_+^γ .

Moreover, since $F(e^{i\pi}|\lambda|, \beta) = e^{i\pi} F(|\lambda|, e^{-i\pi}\beta)$, it follows that the negative zeros of $F(\lambda, \beta)$ are *minus* the positive zeros of $F(\lambda, e^{-i\pi}\beta)$. Consequently,

$$\zeta_-(s, \beta) = e^{-i\pi s} \zeta_+(s, e^{-i\pi}\beta). \tag{B.12}$$

Taking into account that

$$\begin{aligned} \frac{F'(-i\mu, \beta)}{F(-i\mu, \beta)} &= \frac{1 + \mu^2[\psi(\mu^2/2) - \psi(\alpha + \mu^2/2)]}{-i\mu \left[1 - i\beta \frac{\Gamma(\alpha + \mu^2/2)}{\mu\Gamma(\mu^2/2)} \right]} \\ -i\mu\psi(\mu^2/2) &= i[\Delta_1(\mu) + \Delta_2(\mu, \beta)] + O(\mu^{-3}), \end{aligned} \tag{B.13}$$

with

$$\Delta_1(\mu) = -\mu \log\left(\frac{\mu^2}{2}\right) + \frac{1}{\mu}, \quad \Delta_2(\mu, \beta) = \frac{2g}{\mu \left[1 - i\frac{\beta}{\mu} \left(\frac{\mu^2}{2}\right)^{-g+1/2} \right]}, \tag{B.14}$$

we see the right-hand side of equation (B.11) converges to an analytic function on the open half-plane $\text{Re}(s) > 2$, region from which it can be meromorphically extended to the left.

For example, taking into account that

$$\frac{F'(-ie^{i\pi}\mu, \beta)}{F(-ie^{i\pi}\mu, \beta)} = e^{i\pi} \frac{F'(-i\mu, e^{-i\pi}\beta)}{F(-i\mu, e^{-i\pi}\beta)}, \tag{B.15}$$

we can write

$$\begin{aligned}
 -2\pi \zeta_+(s, \beta) &= -2 \sin\left(\frac{\pi s}{2}\right) \int_1^\infty \mu^{-s} \Delta_1(\mu) d\mu \\
 &+ i \int_1^\infty \mu^{-s} \left\{ e^{i\pi s/2} \Delta_2(\mu, \beta) - e^{-i\pi s/2} \Delta_2(\mu, e^{-i\pi} \beta) \right\} d\mu \\
 &+ e^{i\pi s/2} \int_1^\infty \mu^{-s} \left\{ \frac{F'(-i\mu, \beta)}{F(-i\mu, \beta)} - i[\Delta_1(\mu) + \Delta_2(\mu, \beta)] \right\} d\mu \\
 &- e^{-i\pi s/2} \int_1^\infty \mu^{-s} \left\{ \frac{F'(-i\mu, e^{-i\pi} \beta)}{F(-i\mu, e^{-i\pi} \beta)} - i[\Delta_1(\mu) + \Delta_2(\mu, e^{-i\pi} \beta)] \right\} d\mu \\
 &+ e^{i\pi s/2} \int_{e^{i\pi}}^1 \mu^{-s} \frac{F'(-i\mu, \beta)}{F(-i\mu, \beta)} d\mu, \tag{B.16}
 \end{aligned}$$

where the first integral on the right-hand side converges for $\text{Re}(s) > 2$, the second one converges for $\text{Re}(s) > 0$, the third and fourth ones exist for $\text{Re}(s) > -2$, and the fifth one (evaluated on a curve going from -1 to 1 on the upper open half-plane, near the real axis) is an entire function of s .

For the analytic extension of the first term on the right-hand side of equation (B.16) we have

$$\begin{aligned}
 I_1(s) &= -2 \sin\left(\frac{\pi s}{2}\right) \int_1^\infty \mu^{-s} \Delta_1(\mu) d\mu \\
 &= -2 \sin(\pi s/2) \left[\frac{1}{s} - \frac{2}{(s-2)^2} + \frac{\log(2)}{s-2} \right], \tag{B.17}
 \end{aligned}$$

and for the second one (calling $x = \mu^{-2g}$)

$$\begin{aligned}
 I_2(s) &= \text{Re} \left\{ 2i e^{i\pi s/2} \int_1^\infty \mu^{-s} \Delta_2(\mu, \beta) d\mu \right\} \\
 &= -\text{Re} \left\{ 2i e^{i\pi s/2} \lim_{\mu \rightarrow \infty} \int_1^{\mu^{-2g}} \frac{x^{\frac{s}{2g}-1} dx}{1 - i 2^{g-\frac{1}{2}} \beta x} \right\}, \tag{B.18}
 \end{aligned}$$

for $g \neq 0$, while $I_2(s) \equiv 0$ for $g = 0$.

According to the sign of g , we straightforwardly get:

- For $g > 0$,

$$\begin{aligned}
 I_2(s) &= -\frac{4g}{s} \sin\left(\frac{\pi s}{2}\right) - \frac{2^{g+3/2} g \beta}{s+2g} \cos\left(\frac{\pi s}{2}\right) \\
 &+ 2^{2g} \beta^2 \int_0^1 x^{\frac{s}{2g}+1} \frac{\sin\left(\frac{\pi s}{2}\right) + 2^{g-1/2} \beta x \cos\left(\frac{\pi s}{2}\right)}{1 + 2^{2g-1} \beta^2 x^2} dx, \tag{B.19}
 \end{aligned}$$

where the last integral converges for $s > -4g$. Note the pole² at $s = -2g$.

² This singularity implies that the ζ -function of Q_+^γ ,

$$\zeta(s, \beta) \equiv \zeta_+(s, \beta) + \zeta_-(s, \beta) = \zeta_+(s, \beta) + e^{-i\pi s} \zeta_+(s, e^{-i\pi} \beta) \tag{B.20}$$

presents a simple pole at $s = -2g$,

$$\zeta(s, \beta) = \frac{2^{g+3/2} (e^{2i\pi g} - 1) g \beta \cos(g\pi)}{s+2g} + O(s+2g)^0. \tag{B.21}$$

The residue, which depends on the self-adjoint extension through β , vanishes only for the $g = 0$ case, and for $\beta = 0$ (with any value of $g \in (-1/2, 1/2)$). This is another example of a singular potential leading to self-adjoint extensions with associated ζ -functions presenting poles at positions which do not depend only on the order of the differential operator and the dimension of the manifold, as is the general rule valid for the case of smooth coefficients (see [32–34]).

- For $g < 0$ and $\beta \neq 0$,

$$I_2(s) = -\frac{2^{-g+5/2}g}{\beta(s-2g)} \cos\left(\frac{\pi s}{2}\right) + \int_1^\infty x^{\frac{s}{2g}-2} \frac{2\beta x \sin\left(\frac{\pi s}{2}\right) - 2^{-g+3/2} \cos\left(\frac{\pi s}{2}\right)}{\beta[1+2^{2g-1}\beta^2 x^2]} dx, \quad (\text{B.22})$$

where the last integral converges for $s > 4g = -4|g|$. Note the pole at $s = 2g = -|2g|$.

Note that $\zeta_+(s, \beta)$ is analytic in a neighbourhood of the origin. From equations (B.16), (B.17), (B.19) and (B.22) it is easy to get the first term of the Taylor expansion of $\zeta_+(s, \beta)$ around $s = 0$,

$$-2\pi \zeta_+(s \approx 0, \beta) = -\pi + \begin{cases} -2\pi g, & g > 0 \\ 0, & g \leq 0 \end{cases} + \int_1^\infty \left[\frac{F'(-i\mu, \beta)}{F(-i\mu, \beta)} - \frac{F'(-i\mu, e^{-i\pi}\beta)}{F(-i\mu, e^{-i\pi}\beta)} \right] d\mu \\ + i[\log F(-i, \beta) - \log F(i, \beta)] + O(s), \quad (\text{B.23})$$

where the remaining integral can be evaluated taking into account that

$$\frac{\Gamma\left(\frac{1}{2} - g + \frac{\mu^2}{2}\right)}{\mu \Gamma\left(\frac{\mu^2}{2}\right)} = 2^{g-1/2} \mu^{-2g} \{1 + O(\mu^{-2})\}. \quad (\text{B.24})$$

We get

$$\zeta_+(s=0, \beta) = \begin{cases} g, & 0 < g < 1/2, \\ -\frac{1}{\pi} \arctan\left(\frac{\beta}{\sqrt{2}}\right), & g = 0, \\ -\frac{1}{2} \text{sign}(\beta), & -1/2 < g < 0. \end{cases} \quad (\text{B.25})$$

Therefore, from equations (B.8), (B.12) and (B.25), it is straightforward to get for the spectral asymmetry of Q_+^γ at $s = 0$

$$\eta(s=0) = [\zeta_+(s, \beta) - \zeta_+(s, e^{-i\pi}\beta)]|_{s=0} \\ = \begin{cases} 0, & 0 < g < 1/2, \forall \beta, \\ -\frac{2}{\pi} \arctan\left(\frac{\beta}{\sqrt{2}}\right), & g = 0, \quad \forall \beta \in \mathbb{R}^+, \\ -\text{sign}(\beta), & -1/2 < g < 0, \quad \forall \beta \neq 0, -\infty. \end{cases} \quad (\text{B.26})$$

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